Advanced Systems Theory

20/01/2026, Tuesday, 11:45 – 13:45

1 Disturbance decoupling problem (DDP) (10 + 5 + 5 + 5 = 25 pts)

Consider the control system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed, \\ z &= Hx \end{aligned}$$

with

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

(a) Compute $\mathcal{V}^*(\ker H)$.

(b) Show that DDP (with state feedback) is solvable for this system.

(c) Show that $u = Fx$ with $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ solves the DDP if and only if $f_3 = -1$.

(d) Does there exist a feedback gain F that solves the DDP and renders $A + BF$ stable?

SOLUTION:

(a): To compute $\mathcal{V}^*(\ker H)$, we can use the following subspace algorithm:

$$\mathcal{V}_0 = \ker H \quad \text{and} \quad \mathcal{V}_{k+1} = \ker H \cap A^{-1}(\mathcal{V}_k + \operatorname{im} B) \text{ for } k \geqslant 0.$$

Note that

- $\mathcal{V}_0 = \ker H = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.
- $\mathcal{V}_0 + \operatorname{im} B = \ker H = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ since $\operatorname{im} B \subseteq \ker H$.
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in A^{-1}(\mathcal{V}_0 + \operatorname{im} B)$ if and only if $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + c \\ a \\ b \end{bmatrix}$, we see that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in A^{-1}(\mathcal{V}_0 + \operatorname{im} B)$ if and only if $a = 0$. Therefore, $A^{-1}(\mathcal{V}_0 + \operatorname{im} B) = \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- $\mathcal{V}_1 = \ker H \cap A^{-1}(\mathcal{V}_0 + \operatorname{im} B) = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- $\mathcal{V}_1 + \operatorname{im} B = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \operatorname{im} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \operatorname{im} \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{V}_0 + \operatorname{im} B$.
- $\mathcal{V}_2 = \ker H \cap A^{-1}(\mathcal{V}_1 + \operatorname{im} B) = \ker H \cap A^{-1}(\mathcal{V}_0 + \operatorname{im} B) = \mathcal{V}_1 = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, we can conclude that $\mathcal{V}^*(\ker H) = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b): The DDP is solvable if and only if $\operatorname{im} E \subseteq \mathcal{V}^*(\ker H)$. Since $\operatorname{im} E = \operatorname{im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathcal{V}^*(\ker H)$, we can conclude that the DPP is solvable.

(c): The feedback $u = Fx$ solves the DDP if and only if F is a 'friend' of $\mathcal{V}^*(\ker H)$, that is, $(A + BF)\mathcal{V}^*(\ker H) \subseteq \mathcal{V}^*(\ker H)$. By taking $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$, note that

$$A + BF = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \begin{bmatrix} 2 + f_1 & f_2 & 1 + f_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since

$$(A + BF)\mathcal{V}^*(\ker H) = \operatorname{im} \begin{bmatrix} 1 + f_3 \\ 0 \\ 0 \end{bmatrix},$$

we see that $(A + BF)\mathcal{V}^*(\ker H) \subseteq \mathcal{V}^*(\ker H)$ if and only if $f_3 = -1$.

(d): From (c), we see that if the feedback $u = Fx$ with $F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ solves the DDP, then

$$A + BF = \begin{bmatrix} 2 + f_1 & f_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Since the last column of $A + BF$ is a zero column, $\det(A + BF) = 0$ and hence $A + BF$ has a zero eigenvalue. Therefore, we can conclude that there is no F that solves the DDP and renders $A + BF$ stable.

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2 Stabilization algorithm by Bass (2 + 5 + 2 + 1 + 10 = 20 pts)

Suppose that the pair (A, B) is stabilizable.

(a) Show that there exists $\alpha > 0$ such that $-\alpha I - A$ is stable.

(b) Explain why there exists a unique $P \geqslant 0$ such that $(\alpha I + A)P + P(\alpha I + A^T) = 2BB^T$.

(c) Show that $\ker P \subseteq \ker B^T$.

(d) Show that there exists F such that $FP = -B^T$.

(e) Show that $A + BF$ is stable for all F such that $FP = -B^T$.

(a): Note that $\lambda \in \sigma(A)$ if and only if $-\alpha - \lambda \in \sigma(-\alpha I - A)$. Let $\beta = \max\{-\operatorname{Re}(\lambda) \mid \lambda \in \sigma(A)\}$. Note that $-\alpha - \operatorname{Re}(\lambda) < 0$ for every $\lambda \in \sigma(A)$ if $\alpha > \max(0, \beta)$.

(b): Since $-\alpha I - A$ is stable, there exists a unique $P \geqslant 0$ such that $(-\alpha I - A)P + P(-\alpha I - A)^T = -2BB^T$. This show that there exists a unique $P \geqslant 0$ such that $(\alpha I + A)P + (\alpha I + A)^T = 2BB^T$.

(c): Let $\xi \ker P$. Note that

$$2\xi^T BB^T \xi = \xi^T \left((\alpha I + A)P + P(\alpha I + A)^T\right) \xi = 0.$$

From this observation, we can conclude that $B^T \xi = 0$. Thus, we proved that $\ker P \subseteq \ker B^T$.

(d): The inclusion $\ker P \subseteq \ker B^T$ is equivalent to $\operatorname{im} B \subseteq \operatorname{im} P$. Therefore, there exists G such that $B = PG$. As such, there exists F such that $FP = -B^T$.

(e): Let F be such that $FP = -B^T$. Note that

$$(A + BF)P + P(A + BF^T) = AP + PA^T - 2BB^T. \tag{1}$$

Since $(\alpha I + A)P + (\alpha I + A)^T = 2BB^T$, we see that

$$AP + PA^T = 2BB^T - 2\alpha P. \tag{2}$$

It follows from (1) and (2) that

$$(A + BF)P + P(A + BF^T) = -2\alpha P. \tag{3}$$

Now, let (λ, x) be an eigenpair of $(A + BF)^T$, that is $x^*(A + BF) = \lambda x^*$ and $x \neq 0$. From (3), we have

$$2\operatorname{Re}(\lambda) x^* x = x^* \left((A + BF)P + P(A + BF^T)\right) x = -2x^* P x.$$

In case $x^* P x > 0$, we can conclude that $\operatorname{Re}(\lambda) < 0$. In case $x^* P x = 0$, we have $x \in \ker P$. From (c), we see that $x \in \ker B^T$. Therefore, $\lambda x^* = x^*(A + BF) = x^* A$. This means that λ is an uncontrollable eigenvalue of the pair (A, B). Since this pair is stabilizable, we see that $\operatorname{Re}(\lambda) < 0$. The positive definiteness of P implies that either $x^* P x > 0$ or $x^* P x = 0$. Therefore, we prove that $A + BF$ is stable.

3 Algebraic Riccati equation (20 pts)

Consider the algebraic Riccati equation $A^T P + PA - PBB^T P + Q = 0$ where P and Q are symmetric matrices. Show that $A - BB^T P$ is stable if P, Q are positive semidefinite and (Q, A) is detectable.

Suppose that P, Q are positive semidefinite and (Q, A) is detectable. From the algebraic Riccati equation, we have

$$(A - BB^T P)^T P + P(A - BB^T P) = -PBB^T P - Q. \tag{4}$$

Let (λ, x) be an eigenpair of $A - BB^T P$, that is $(A - BB^T P)x = \lambda x$ and $x \neq 0$. By pre- and post-multiplying (4) by x^* and x, respectively, we obtain

$$2\,\mathrm{Re}(\lambda) x^* P x = -x^* PBB^T P x - x^* Q x =: a.$$

Note that $a \leqslant 0$. Suppose, first, that $a < 0$. Since P is positive semidefinite, we can conclude in this case that $x^* P x > 0$ and hence $\mathrm{Re}(\lambda) < 0$. Suppose, now, that $a = 0$. Then, we have $x^* PBB^T P x = 0$ and $x^* Q x = 0$. Since both $PBB^T P$ and Q are positive semidefinite, we see that $B^T P x = 0$ and $Qx = 0$. Therefore, we have $\lambda x = (A - BB^T P)x = Ax$ and $Qx = 0$. Since (Q, A) is detectable, this implies that $\mathrm{Re}(\lambda) < 0$. Consequently, $A - BB^T P$ is stable.

4 Data-driven control (3 + 2 + 8 + 6 + 6 = 25 pts)

Consider the true unknown system

$$\boldsymbol{x}(t+1) = A_{\text{true}}\boldsymbol{x}(t) + B_{\text{true}}\boldsymbol{u}(t)$$

where $t \in \mathbb{Z}_+$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{u} \in \mathbb{R}^m$. Suppose that the data

$$U_- = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}$$

are harvested from the true system. Define

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \quad \text{and} \quad X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

Also, define the set of all data-generating systems

$$\Sigma := \{(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \mid X_+ = AX_- + BU_-\}.$$

The data (U_-, X) are called *informative for stabilization by state feedback* if there exists K such that $A + BK$ is stable for all $(A, B) \in \Sigma$. We have proven, in the lectures, that the data are informative in this sense if and only if X_- has full row rank and there exists **a** right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable. Moreover, if these conditions are satisfied then $K = U_-X_-^\dagger$ stabilizes every $(A, B) \in \Sigma$. Checking whether X_- has full row rank is easy. If this condition is satisfied, however, there are infinitely many right inverses in general. As such, verifying the second condition is not straighforward. In this problem, we want to device an algorithm to compute a stabilizing gain from informative data.

Suppose that the data (U_-, X) are informative for stabilization by state feedback. Then, X_- has full row rank. By the rank-nullity theorem, the dimension of $\ker X_-$ is $T - n$. Let $V \in \mathbb{R}^{T \times (T-n)}$ be a matrix whose columns form a basis for $\ker X_-$.

(a) Show that $X_-X_-^T$ is nonsingular.

(b) Let $W = X_-^T(X_-X_-^T)^{-1}$. Show that W is a right inverse of X_-.

(c) Show that $X_-^\dagger$ is a right inverse of X_- if and only if $X_-^\dagger = W + VF$ for some $F \in \mathbb{R}^{(T-n)\times n}$.

(d) Show that (X_+W, X_+V) is stabilizable.

(e) Show that $K = U_-(W + VF)$ stabilizes every $(A, B) \in \Sigma$ if F is such that $X_+W + X_+VF$ is stable.

(a): Let $\xi \in \ker X_-X_-^T$. Note that $\xi^T X_-X_-^T\xi = 0$. Since $X_-X_-^T \geqslant 0$, we see that $X_-^T\xi = 0$ or equivalently $\xi^T X_- = 0$. This can happen only if $\xi - 0$ as the matrix X_- has full row rank. Therefore, $\ker X_-X_-^T = \{0\}$ and hence $\ker X_-X_-^T$ is nonsingular.

(b): Note that $X_-W = X_-X_-^T(X_-X_-^T)^{-1} = I$. Therefore, W is a right inverse of X_-.

(c): For the 'only if' part, suppose that $X_-^\dagger$ is a right inverse of X_-. Then, we have

$$X_-X_-^\dagger = I. \tag{5}$$

From (b), we have $X_-W = I$. Then, it follows from (5) that $X_-(X_-^\dagger - W) = 0$. In other words, $\operatorname{im}(X_-^\dagger - W) \subseteq \ker X_-$. Since $\ker X_- = \operatorname{im} V$, we see that there exists F such that $X_-^\dagger - W = VF$. Clearly, $X_-^\dagger = W + VF$.

For the 'if' part, let $X_-^\dagger = W + VF$ for some F. Note that $X_-X_-^\dagger = X_-(W + VF) = I$ since W is a right inverse of X_- and $\ker X_- = \operatorname{im} V$. Therefore, $X_-^\dagger = W + VF$ is a right inverse of X_-.

(d): From the informativity, we know that there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable. From (c), we know that there exists F such that $X_-^\dagger = W + VF$. Therefore, $X_+(W + VF) = X_+W + X_+VF$ is stable. This means that (X_+W, X_+V) is stabilizable.

(e): If $X_+W + X_+VF$ is stable for some F, then it follows from (c) and (d) that $W + VF$ is a right inverse of X_- such that $X_+(W + VF)$ is stable. This means that $K = U_-(W + VF)$ stabilizes every $(A, B) \in \Sigma$.